Equisingular Deformations of Plane Curve and of Sandwiched Singularities

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Abstract

Let C be an isolated plane curve singularity, and (C, l) be a decorated curve. In this article we compare the equisingular deformations of C and the sandwiched singularity $X(C, l)$. We will prove that for $l \gg 0$ the functor of equisingular deformations of C and (C, l) are equivalent. From this we deduce a proof of a formula for the dimension of the equisingular stratum. Furthermore we will show how compute the equisingularity ideal of the curve singularity C, given the minimal (good) resolution of C.

1 Introduction

In the sixties, Zariski [20], [21], [22] started the modern study of equisingularity of plane curve singularities. Two plane curve singularities are called equisingular if one can simultaneously resolve their singularities. Zariski proved that equisingular plane curve singularities are topologically equivalent. In particular, the characteristic exponents of the branches are the same, and corresponding intersection numbers are the same. For more characterizations, we refer to the article of Teissier, [15].

Infinitesimal equisingular families of plane curve singularities were studied by Wahl, [16]. He defined the functor of Artin rings of equisingular deformations of plane curve singularities, and showed that it is a smooth subfunctor of deformations of the plane curve singularity itself. He showed that the tangent space can be identified with an ideal in the local ring of the curve: the so-called equisingularity ideal. For an irreducible curve singularity, it seems, given a parametrization, not to be to difficult to compute this equisingularity ideal. see [15].

In the fourth section of this article we will give an (inductive) algorithm for computing the equisingularity ideal, given a minimal good resolution of the curve singularity. The idea is to look at a *simultaneous resolution* functor. This is a functor (of Artin rings) which describes all deformations of the plane curve singularity (with sections) which can be simultaneously resolved. If one considers such deformations over smooth curves, the singularity is allowed to split up in several singularities. For the equisingular deformation functor, this is not

allowed. The tangent space of this simultaneous resolution functor can be calculated inductively. An infinitesimal deformation with simultaneous resolution in particular gives an infinitesimal deformation of the minimal embedded good resolution of C. In particular we get for each exceptional curve a deformation of a small neighborhood of this curve. If all these deformations are trivial we have that are deformation is equisingular, by using results of Wahl [16]. Thus the equisingularity ideal is the kernel of a map φ from the tangent space of the functor of deformations with simultaneous resolution to a direct sum ofcohomology groups $\oplus H^1(F_i, N_{F_i/Z})$. Here Z is the minimal good resolution, and the F_i are the exceptional curves of Z.

The study of equisingular deformations of *surface singularities* was started by Wahl in [17]. This theory is more difficult than for curve singularities. For rational surface singularities, however, one still has a good theory. In this paper we study equisingular deformations of a special, but nevertheless rather broad, class of rational surface singularities: the sandwiched singularities. These surface singularities were studied by Zariski [19], Lipman [11], Hironaka [6], Spivakovsky [14], De Jong and Van Straten [10] and Gustavsen [4]. These sandwiched singularities can be constructed by so–called decorated curves (C, l) see [10], so are of type $X(C, l)$. To each branch of C one assigns a natural number satisfying a technical condition. We first show that the analytic type of C is uniquely determined by the analytic type of $X(C, l)$, as soon as $l \gg 0$. Then we will show that for $l \gg 0$ the equisingular deformations of C are in one-one correspondence with equisingular deformations of the sandwiched singularity $X(C, l)$. Using two different formulas for the dimension of the Artin component of $X(C, l)$, we are then able to deduce a formula for the dimension of the equisingular stratum of C.

2 Equisingular Deformations of Sandwiched Singularities.

We will consider *sandwiched* singularities. First consider a curve singularity $C = \cup_{i \in T} C_i$. We let $m(i)$ be the sum of the multiplicities of branch i in the multiplicity sequence of the minimal embedded resolution of C_i, and $M(i)$ be the sum of the multiplicities of branch i in the multiplicity sequence of the minimal good embedded resolution of C_i. Let $\mathrm{eb}(C)$ be the number of extra blowing-ups needed to come from a *minimal embedded* resolution of C to a *minimal* **good** *embedded* resolution of C. Note the following:

Lemma 2.1.

$$\sum_{i \in T} M(i) - m(i) = \mathrm{eb}(C).$$

Let $l : T \longrightarrow \mathbb{N}$ be such that $l(i) \geq M(i) + 1$ for all i. The pair (C, l) is called a *decorated curve*, see [10], 1.3. One gets the *sandwiched singularity* $X(C, l)$ as follows. First take a minimal good embedded resolution of C, and

then do $l(i) - M(i)$ consecutive blowing-ups at the strict transform of the i'th branch, thereby inducing a chain of $l(i) - M(i)$ new exceptional curves for each i. We get a modification of $\mathbb{C}^2$:

$$(Z(C,l), F) \longrightarrow (\mathbb{C}^2, 0).$$

Consider E, the subgraph of all irreducible components whose self-intersection is not -1. From the resolution process one sees that E is connected, has negative definite intersection matrix, and so can be contracted by the result of Grauert–Mumford to a normal surface singularity $X(C,l)$, which one easily sees to be rational. This singularity is denoted by $X(C,l)$. The modification $(Z(C,l), F)$ we can get by blowing up a *complete ideal* $I(C,l)$.

The following theorem was proved by Gustavsen [4] for the case the plane curve singularities are irreducible.

Theorem 2.2. *Consider decorated curves (C,l) and (C',l'). Suppose that the surface singularities $X(C,l)$ and $X(C',l')$ are isomorphic. Suppose $l \gg 0$ and $l' \gg 0$. Then C and C' are isomorphic and $l = l'$. (This is to be interpreted that for some isomorphism corresponding branches have the same number attached.)*

Proof. Let φ be an isomorphism taking $X(C',l')$ to $X(C,l)$. By the universal property of blowing–up, this extends to an isomorphism between the minimal resolutions $\varphi : (\widetilde{X}(C',l'), E') \longrightarrow (\widetilde{X}(C,l), E)$. In particular the dual graphs of $X(C,l)$ and $X(C',l')$ are isomorphic. We write $C = \cup_{i \in T} C_i$. As both $l \gg 0$ and $l' \gg 0$ there are exactly $\#T$ very long chains of (-2)–curves in both E and E'. Due to their length, they can be recognized. To each endpoint G'_i of such a very long chain in E' there is, by the construction of sandwiched singularities, a (-1)–curve F'_i in $Z(C',l')$ and a smooth curve C'_i intersecting F'_i. Similar for $Z(C,l)$. The isomorphism φ sends G'_i to a G_i, as the resolution graphs of $X(C',l')$ and $X(C,l)$ are isomorphic. By using the isomorphism φ, we may glue every F'_i (and in it C'_i) to the curve G_i in $X(C,l)$. Thus we may assume that the two resolutions are equal: $(Z(C',l'), E') = (Z(C,l), E)$.

Without loss of generality, we may assume that both x and y are generic elements of $\mathcal{O}_{C'}$. By looking at the image under φ of the divisors of the pull-back of (x) and (y) on $X(C',l')$ (which is the fundamental cycle of $Z(C'l')$ restricted to E' plus a non-compact curve intersecting the first blown up curve), we see that the divisors of $\varphi(x)$ and $\varphi(y)$ are the divisors of two functions which generate the maximal ideal of $\mathcal{O}_C$. Thus φ maps C' to an isomorphic curve, which except for the (-1)–curves has the same resolution as C. Thus we may and will suppose that the resolutions $Z(C',l')$ and $Z(C,l)$ are equal. From the construction of the $Z(C,l)$ out of the $\widetilde{X}(C,l)$ it follows that $l = l'$.

There is an algorithm to get, from the resolution of a plane curve singularity, the resolution of every irreducible component. Following this algorithm, we see that in the resolution of any irreducible component C_i of C there is a very long chain of (-2)–curves. By the remarks above this is, except for the (-1)–curves, also the resolution for C'_i. By the formula for the intersection number,

see, for example, [7] Theorem 5.4.8, the intersection number between C and C' increases if the chains of (-2) curves, and thus $l(i)$ and $l'(i)$ become bigger. This intersection number is equal to both $\dim \mathbb{C}\{x,y\}/(f_i, f_i')$, and also to the vanishing order of f_i' on $\widetilde{C_i}$, where $\widetilde{C_i}$ is the normalization of C_i. Here the curves C_i and C_i' are defined by irreducible f_i and f_i'. Thus we may assume that for all i the vanishing order of f_i' on $\widetilde{C_i}$ is at least $k \cdot c(i)$, for some large k, and where $c(i)$ is the i'th conductor number of C, corresponding to the branch C_i. By definition of the conductor, every function which vanishes with order at least $c(i)$ on the normalization $\widetilde{C_i}$ for all i, is an element of the maximal ideal (x,y) of $\mathcal{O}_C = \mathbb{C}\{x,y\}/(f)$. It follows that the class of $f' = \prod f_i'$ in $\mathcal{O}_C$ lies in $(x,y)^k$. Thus $uf - f' \in (x,y)^k$, where we now view (x,y) as an ideal in $\mathbb{C}\{x,y\}$. By symmetry, u is a unit. By taking k large, we may, by the finite determinacy theorem, assume that uf and f' are right equivalent. In particular, their zero sets C and C' are isomorphic. This is what we had to show. $\square$

We now study equisingular deformations. So we have an functor ES_C of equisingular deformation of the curve singularity C, and the functor $ES_{X(C,l)}$ of equisingular deformations of the sandwiched singularity $X(C,l)$. The dimension of the Zariski tangent spaces we denote by $\mathrm{es}(C)$ and $\mathrm{es}(X(C,l))$. We quote the following result due to Gustavsen, see [4], Theorem 3.3.22.

Theorem 2.3. *There is a natural formally smooth map of functors*

$$\mathrm{ES}_C \longrightarrow \mathrm{ES}_{X(C,l)}\,.$$

This in particular, for any equisingular deformation of $X(C,l)$ we can find an equisingular deformation of C, mapping to it naturally, but this equisingular deformation of C might not be unique, even on tangent spaces. The existence of the map $\mathrm{ES}_C \longrightarrow \mathrm{ES}_{X(C,l)}$ is quite obvious. An equisingular deformation of C induces a deformation of $Z(C,l)$, and thus a deformation of the resolution of $X(C,l)$. This thus gives an equisingular deformation of $X(C,l)$.

Our aim is to prove that for $l \gg 0$, the formally smooth map is in fact an isomorphism. For this, it suffices to show that it is an isomorphism on tangent spaces.

Proposition 2.4. *Let C be an isolated curve singularity, and (C,l) be a decorated curve. Suppose that $l \gg 0$. Then the Zariski tangent spaces of the equisingular deformations of $X(C,l)$ and of C are isomorphic. In particular for the dimensions we have* $\mathrm{es}(X(C,l)) = \mathrm{es}(C)$.

We use the theory developed in [10]. Let C be given by $f = 0$, and $c(i)$ be the conductor number on branch C_i. We consider a function $g(x,y)$ whose vanishing order on the normalization of C_i is equal to $c(i) + l(i)$ for all $i \in T$. In [10] the following non–isolated surface singularity in $\mathbb{C}^3$ is considered:

$$Y(C,l) = V(zf - g).$$

on the choice of g. The normalization of $Y(C,l)$ is proved to be $X(C,l)$. Let $\Sigma \subset \mathbb{C}^2$ be the space defined by the conductor I of C. We will consider R.C.–deformations of the pair (Σ, C). For the definition of R.C.–deformations, see for example [10], Appendix, and the references mentioned there. This functor is canonically isomorphic to deformations of the diagram $\widetilde{C} \longrightarrow C$, where $\widetilde{C} \longrightarrow C$ is the normalization. The space of infinitesimal deformation of R.C.–deformations of the pair (Σ, C), we denote by $T^1(\Sigma, C)$. It was stated, but not properly shown in [10], Remark 3.18, that for $l \gg 0$ we have an exact sequence

$$0 \longrightarrow I^{ev}/(f, \Theta_\Sigma(g)) \longrightarrow T^1_{X(C,l)} \longrightarrow T^1(\Sigma, C) \longrightarrow 0. \tag{1}$$

Here $I^{ev} = \{g \in \mathbb{C}\{x,y\} : \mathrm{ord}(g_{|\widetilde{C}_i}) \geq l(i) + m(i) \text{ for all } i\}$ for $\widetilde{C}_i \longrightarrow C_i$ the normalization. Furthermore Θ_Σ are all derivations θ of $\mathbb{C}\{x,y\}$ satisfying the two conditions $\theta(f) \subset (f)$ and $\theta(I) \subset I$. If t_i is a parameter on $\widetilde{C}_i$, one can identify Θ_Σ with the module generated by the $t_i \frac{\partial}{\partial t_i}$. Let us see how the sequence 1 comes about. In [10], Proposition 3.7 it is shown that every infinitesimal deformation of $X(C,l)$ can be obtained, up to isomorphism, as follows. Take an infinitesimal R.C.–deformation $(\Sigma_\varepsilon, C_\varepsilon)$ of (Σ, C), and a g_ε such that also $(\Sigma_\varepsilon, (g_\varepsilon = 0))$ is an R.C–deformation of $(\Sigma, (g = 0))$. Suppose that Σ_ε is defined by the ideal I_ε, and C_ε by $f_\varepsilon = 0$. We get a space Y_ε defined by $zf_\varepsilon - g_\varepsilon = 0$, and the space X_ε is defined by the local ring

$$\mathrm{Hom}_{Y_\varepsilon}(I_\varepsilon, I_\varepsilon).$$

So X_ε is obtained by "simultaneous normalization". In general, the choice of the isomorphism class of the R.C.–deformation of (Σ, C) is not uniquely determined by the given infinitesimal deformation of $X(C,l)$. But we will see and have to show that in case $l \gg 0$ it in fact does. By linearity, it suffices to show that if we have a trivial infinitesimal deformation of $X(C,l)$, we are only allowed to deform (Σ, C) trivially. Now a trivial first order family can be extended to a trivial family over a germ of a smooth curve T, and we may assume that this family is a product family. By the results of [10], in particular Theorem 3.3 we can therefore extend the R.C–deformation $(\Sigma_\varepsilon, C_\varepsilon)$ and $(\Sigma_\varepsilon, (g_\varepsilon = 0))$ to deformations over this smooth curve. Thus we get an I_T defining Σ_T, f_T defining C_T, and some g_T, such that the local ring of the trivial family is given by $\mathrm{Hom}_{Y_T}(I_T, I_T)$, where Y_T is given by $zf_T - g_T = 0$.

Writing $F_t = \sum_i t^i f_i$, we have that for all small t the zero set of F_t intersects on $Z(C,l)$ the (-1)–curve. This is because the induced deformation of $X(C,l)$, and thus of the resolution, is a product family. For fixed k, it follows, as in the proof of Theorem 2.2 that by taking $l \gg 0$, we may assume that $f_i \in (x,y)^{k+1}$ for all i. We take k so big, that $(x,y)^{k+1} \subset (x,y)^2\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. We write $x_1 = x$, and $x_2 = y$. Thus we can write, formally, $\sum_i t^{i-1} f_i = \sum_i \xi_i(t) \frac{\partial f}{\partial x_i}$, with $\xi_i(t) \in (x,y)^2$. Hence

$$\frac{\partial F_T}{\partial t} = \sum_i t^{i-1} f_i = \sum_i \xi_i(t) \frac{\partial F_T}{\partial x_i} - \sum_j \sum_i \xi_i(t) t^j \frac{\partial f_j}{\partial x_i}.$$

Note that $\sum_j \sum_i \xi_i(t) t^j \frac{\partial f_j}{\partial x_i} \in (x,y)^{k+2}$. Iterating this procedure we get

$$\frac{\partial F_T}{\partial t} = \sum \overline{\xi}_i \frac{\partial F_T}{\partial x_i}.$$

for some formal $\overline{\xi}_i(t) \in (x,y)^2$. By Artin's Approximation Theorem, [1] we may even assume that the ξ_i are analytic. By the characterization of Local Analytic Triviality, see for example [7], Theorem 9.1.7, it follows that the deformation of C_T, given by $F_T = 0$, is trivial. By looking at the conductor, it follows that the family Σ_T is trivial. In particular, the induced first order R.C.–deformation of (Σ, C) is trivial. Hence we get a map

$$T^1_{X(C,l)} \longrightarrow T^1(\Sigma, C),$$

which one easily sees to be linear. As $l \gg 0$, it is also surjective, see [10], Proof of Theorem 3.3. The kernel one gets, as described in [10], as follows. Consider elements $g' \in \mathbb{C}\{x,y\}$ such that $g_\varepsilon := g + \varepsilon g'$ give an R.C–deformation of $(\Sigma, (g = 0))$, by keeping Σ fixed. As proved in [10], these are exactly the elements in I^{ev}. Then consider Y_ε defined by $zf - g_\varepsilon = 0$. We get X_ε as above. Obviously the g' in $\Theta_\Sigma(g)$ give trivial deformations Y_ε of Y, and thus trivial deformations X_ε of X. As soon as C is singular, and the class of g' in $I^{ev}/(f, \Theta_\Sigma(g))$ is nonzero, this deformation is non-trivial. Then if it where, the deformation Y_ε could be extended to a trivial family Y_T given by $zf - g_T = 0$. But as the vanishing order of g' on the normalization $\widetilde{C}_i$ of at least one branch C_i at the point mapping to the singular point of C is strictly smaller that $c(i)+l(i)$, this then also holds for a general g_t for t small. Thus the resolution graph of the normalization of $zf - g_t$ is different from the resolution graph of $X(C,l)$. In particular the deformation is not trivial, contradiction. Thus we proved the existence of the exact sequence (1).

Proof of Proposition 2.4. The proposition follows from the exact sequence (1). □

Remark 2.5. Given the decorated curve (C,l), the kernel of the map

$$\mathrm{ES}_C(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{ES}_{X(C,l)}(\mathbb{C}[\varepsilon])$$

can be computed explicitly. Namely, one searches for equisingular deformations of C, which deform $X(C,l)$ trivially. Given the projection $Y(C,l)$, say given by $zf - g = 0$, one can compute all R.C.–deformations of $Y(C,l)$ which deform $X(C,l)$ trivially. Consider generators $u_0 = 1, u_1, \dots, u_k$ of $\mathcal{O}_{X(C,l)}$ as $\mathcal{O}_{Y(C,l)}$–module. In [8], and [10], A.9, for any vector field θ on $\mathbb{C}^3$ an action of $u_i\theta$ on $zf - g$ is defined, which, by simultaneously normalizing, give all infinitesimal trivial deformations of $X(C,l)$. These deformations in general, do not keep the form $zf - g$ fixed, that is, they are in general not of this the type $zf_\varepsilon - g_\varepsilon = 0$, for some deformation f_ε and g_ε. But we can look at the subspace of deformations that do! Thus we get can compute all f' so that there exist a g' so

that $z(f+\varepsilon f')-(g+\varepsilon g')$ that gives an R.C.–deformation of $Y(C,l)$ which gives trivial deformation of $X(C,l)$. These give equisingular deformations $f+\varepsilon f'$ of C. The totality of such f' build an ideal in $\mathrm{ES}(\mathbb{C}[\varepsilon])$ giving the kernel of the map $\mathrm{ES}_C(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{ES}_{X(C,l)}(\mathbb{C}[\varepsilon])$.

We consider an example. For C we take four lines through the origin, and for l we take the function assigning two to each branch. The sandwiched singularity $X(C,l)$ is isomorphic to the cone over the rational normal curve of degree 5. As this singularity is taut, we get $\mathrm{es}(X(C,l))=0$. $Y(C,l)$ can be given by $z(y^4-x^4)-x^5=0$. As described in [8], the generators of $\mathcal{O}_{X(C,l)}$ as $\mathcal{O}_{Y(C,l)}$–module, say $1,u_1,u_2,u_3$, correspond to the rows of the matrix:

$$\begin{pmatrix} zy & 0 & 0 & zx+x^2 \\ x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \end{pmatrix}.$$

Note that the lower three rows gives a resolution of the conductor of C, and that $zf-g=z(y^4-x^4)-x^5=0$ is the determinant of this matrix. The columns give linear equations, so we have four of them:

$$\begin{aligned} L_1: &\quad zy+xu_1=0 \\ L_2: &\quad yu_1+xu_2=0 \\ L_3: &\quad yu_2+xu_3=0 \\ L_4: &\quad (zx+x^2)+yu_3=0. \end{aligned}$$

From these, the quadratic equations can be calculated:

$$\begin{aligned} u_1^2 &= zu_2 \\ u_2^2 &= z(z+x) \\ u_3^2 &= (z+x)u_2 \\ u_1u_2 &= zu_3 \\ u_1u_3 &= z(z+x) \\ u_2u_3 &= (z+x)u_1. \end{aligned}$$

Thus we get a total of ten equations, describing the cone over the rational normal curve of degree 5. The actions of $u_i\theta$ on these equations are totally determined by their values on the linear ones, but one needs the quadratic equations to compute them. We do not want to do the whole calculation in detail, but only look at the action of $u_3\frac{\partial}{\partial x}-z\frac{\partial}{\partial y}$. We get the values

$$\begin{aligned} L_1 &\mapsto u_1u_3-z^2=xz \\ L_2 &\mapsto u_2u_3-zu_1=xu_1 \\ L_3 &\mapsto u_2u_3-u_3u_2=0 \\ L_4 &\mapsto zu_2+2xu_2-u_3^2=xu_2. \end{aligned}$$

Thus we get the following infinitesimal deformation of the matrix:

$$\begin{pmatrix} z(y+\varepsilon x) & 0 & 0 & zx+x^2 \\ x & y+\varepsilon x & 0 & 0 \\ 0 & x & y & \varepsilon x \\ 0 & 0 & x & y \end{pmatrix}$$

which gives the R.C.–deformation:

$$z(y^4 - x^4 + \varepsilon x^2 y^2) - x^4.$$

Thus we see that the non–trivial equisingular deformation $y^4 - x^4 + \varepsilon x^2 y^2$ of C maps to the trivial infinitesimal deformation of the cone over the rational normal curve of degree 5.

3 The Dimension of the Artin Component

We quote the following result.

Theorem 3.1 (Wahl, [17], Propositions 2.2 and 2.5). *Let X be a rational surface singularity, $p : \widetilde{X} \longrightarrow X$ be the minimal resolution, E be the exceptional divisor of $p : \widetilde{X} \longrightarrow X$, and $E_1, \dots, E_s$ be the irreducible components of E. Let $b_i = E_i^2$. Then*

$$h^1(\widetilde{X}, \Theta_{\widetilde{X}}) = -\sum_{i=1}^{s}(b_i + 1) + h^1(\widetilde{X}, \Theta(\log E)).$$

The vector space $H^1(\widetilde{X}, \Theta(\log E))$ is the Zariski tangent space to the base space of the equisingular deformations of X. Its dimension we denote by $\mathrm{es}(X)$ for short.

The space $H^1(\widetilde{X}, \Theta_{\widetilde{X}})$ classifies the infinitesimal deformations of the resolution $\widetilde{X}$. Its dimension is the dimension of the Artin component. We will give a different characterization of this dimension for sandwiched singularities $X(C, l)$, for the case $l \gg 0$. Note moreover that we can calculate $I^{ev}/(f, \Theta_\Sigma(g))$ on the normalization, as the conductor I is contained in I^{ev}. Thus we get that

$$\dim_{\mathbb{C}} \left(I^{ev}/(f, \Theta_\Sigma(g)\right) = \sum_{i \in T} \left(l(i) - m(i)\right).$$

Hence from the exact sequence (1) it follows that

Theorem 3.2. *Let (C, l) be a decorated curve, and suppose that $l \gg 0$. Let Σ be the conductor of C. Then*

$$\dim_C(T^1_{X(C,l)}) = \sum_{i \in T} \left(l(i) - m(i)\right) + \dim_{\mathbb{C}} \left(T^1(\Sigma, C)\right).$$

We want to stress that in general for small l the statement of the theorem is false.

We want to understand the deformations on the Artin component of $X(C, l)$. This has been described in [10], 4.13. One can decide whether a deformation occurs on the Artin component by looking at the corresponding R.C.–deformation

of (Σ, C). A general such one–parameter R.C.–deformation of (Σ, C) is one, which has on a general fiber q singular points, where q is the number of infinitely near points of C. For each infinitely near point P of C of multiplicity m_P we have on a general fiber of the deformation a singularity consisting of m_P smooth branches intersecting transversely (which we call ordinary m_p–tuple point). Let A be the closure of the stratum of the base space of R.C.–deformations of (Σ, C) where the above mentioned deformations occur. All elements of I^{ev} are unobstructed against these deformations. This follows immediately from the picture method, see [10]. As the Artin component is smooth, it follows that A is smooth. We get:

Proposition 3.3. *Let (C, l) be a decorated curve with $l \gg 0$. Let A be the stratum described above. Then the dimension of the Artin component of $X(C, l)$ is equal to*

$$\dim(A) + \sum_{i \in T} (l(i) - m(i)) .$$

We now compute $\dim(A)$. The result is:

Proposition 3.4. *Let C be a plane curve singularity, and A be the stratum described above. Then*

$$\dim(A) = \tau(C) - \sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4).$$

Proof. We have an R.C. deformation of (Σ, C) over A. In particular we get a one–parameter deformation of C over A. Let B be the base space of a semi-universal deformation of C. It is a smooth space of dimension $\tau(C)$. By semi-universality we get a map of smooth spaces $A \longrightarrow B$. This map in general is not an immersion. Indeed, Buchweitz [2] showed that the kernel of the map $T^1(\Sigma, C) \longrightarrow T^1_C$ is equal to $m(C) - r(C)$, the multiplicity minus the number of branches. However, for a general point of A we have that for all singularities $m_P(C) = r_P(C)$, so that the map $A \longrightarrow C$ is an immersion at a general point of A. Hence the image of A at a general point is smooth of dimension $\dim(A)$. Thus, by openness of versality, it remains to compute for each infinitely near point P, the codimension of the stratum A of the ordinary m_P–tuple point in the base space of the ordinary m_P–tuple point. This codimension is $\frac{1}{2}(m_P^2 + m_p - 4)$, being two less than the number of monomials of degree smaller than m_p. $\square$

As an application, we can give the dimension of the equisingular stratum of a plane curve singularities. This formula is equivalent to formulas given by Wall [18] and Mattei, [12]. A special case also has been considered in Piene and Kleiman, [13].

Theorem 3.5. *The following formula holds for all plane curve singularities C:*

$$\mathrm{es}(C) = \tau(C) + \mathrm{eb}(C) + \sum_{i=1}^{k} (b_i + 1) - \sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4).$$

where

(1) $\tau(C)$ *is the Tjurina number.*

(2) $\mathrm{IN}(C)$ *is the set of infinitely near points of* C *(including the singularity itself);* m_P *the multiplicity of the infinitely near point* $P \in \mathrm{IN}(C)$*. We only consider infinitely near points of* C *which are singular, that is,* $m_P > 1$*,*

(3) the b_i *are the self-intersection numbers of the exceptional curves* E_i *on a minimal good embedded resolution of* C*.*

Proof. Given C, take a decorated curve (C, l) with $l \gg 0$. Let $E_1, \ldots, E_s$ be the exceptional curves in the minimal resolution of $X(C, l)$. Let $b_i = -E_i^2$. By Wahl's result 3.1, the dimension of the Artin component of $X(C, l)$ is equal to

$$\mathrm{es}(X(C, l)) - \sum_{i=1}^{s} (b_i + 1).$$

On the other hand, combining 3.3 and 3.4 we get that this dimension is also equal to

$$\sum_{i \in T} (l(i) - m(i)) + \tau(C) - \sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4).$$

Using 2.4 we thus get

$$\mathrm{es}(C) = \tau(C) + \sum_{i=1}^{s} (b_i + 1) + \sum_{i \in T} (l(i) - m(i)) - \sum_{P \in \mathrm{IN}(C)} \tfrac{1}{2}(m_P^2 + m_p - 4).$$

It remains to show that $\sum_{i=1}^{s}(b_i + 1) + \sum_{i \in T} (l(i) - m(i)) = \mathrm{eb}(C) + \sum_{i=1}^{k}(b_i + 1)$. This is easy, as we know that we get the minimal resolution of $X(C, l)$ out of the minimal good resolution of C by doing an extra $l(i) - M(i)$ blowing-ups, for each $i \in T$. We thereby introduce a chain of (-2)–curves of length $l(i) - M(i) - 1$, and decrease the selfintersection of the exceptional curve on the minimal resolution of C which C_i intersects by one. From $\mathrm{eb}(C) = \sum_{i \in T} M(i) - m(i)$, see 2.1, the result follows. □

Remark 3.6. It is not so difficult to see that for a decorated curve (C, l) with $l \gg 0$ we have $m(X(C, l)) = m(C) + 1$. Note that we in fact computed the codimension of the Artin component in $T^1_{X(C,l)}$. With a little combinatorics one then proves that the invariant $c(X(C, l))$, introduced by Christophersen and Gustavsen [3], is equal to 0. This can also be proven directly, using the results of [3] and [4], chapter two. In particular, for $l \gg 0$ one gets a formula for the dimension of $T^2_{X(C,l)}$ by using the results of Christophersen and Gustavsen. By using semicontinuity of T^2, this then also holds for all $X(C, l)$ with $m(X(C, l)) = m(C) + 1$. Also, in case $l \gg 0$, one can construct, using the picture method, see [10], a deformation of $X(C, l)$, where on a general fiber we have a cone over

the rational normal curve of degree m_P for each infinitely near point P of C. Using a standard argument, as used in [9], we then get that the obstruction map is surjective, that is, the minimal number of equations for describing the base space of a semi-universal deformation of $X(C,l)$ is equal to the dimension of $T^2_{X(C,l)}$ if $l \gg 0$.

4 Computation of the Equisingularity Ideal

Let C be a plane curve singularity, and let $s : \mathrm{Spec}(\mathbb{C}) \longrightarrow C$ be the singular point of C. We consider the functor of Artin rings $E = E_C$ of deformations of C which are equimultiple along a section, see Wahl [16] 1.3. That is, for an Artinian ring A, an element of $E(A)$ is a pair $C_A \longrightarrow \mathrm{Spec}(A)$, of deformations of C together with a deformation $\overline{s} : \mathrm{Spec}(A) \longrightarrow C_A$ of s such that C_A is equimultiple along $\overline{s}$.

Next Wahl considers the functor $E^{(2)}$. An element of $E^{(2)}(A)$ consists of an element $C_A \longrightarrow \mathrm{Spec}(A)$, $\overline{s} : \mathrm{Spec}(A) \longrightarrow C_A$ of $E(A)$, together with sections $\overline{s}_i : \mathrm{Spec}(A) \longrightarrow B_{\overline{s}}$ for $i = 1, \dots, t$. Here $B_{\overline{s}}$ is the blowing up along the section $\overline{s}$, and $\overline{s}_i : \mathrm{Spec}(A) \longrightarrow B_{\overline{s}}$ should induce equimultiple deformations of all singular points of the *reduced total transform* of C under blowing up the origin in $\mathbb{C}^2$. Inductively one defines the functor $E^{(n)}$, see Wahl [16], 2.7.

We will also consider a slightly different functor $E_{(2)}$. Elements of $E_{(2)}(A)$ are like elements of $E^{(2)}(A)$ except that for non-nodes of the reduced total transform the $\overline{s}_i : \mathrm{Spec}(A) \longrightarrow B_{\overline{s}}$ induce equimultiple deformations of the intersection points of the *strict transform* $\overline{C}$ of C and the exceptional curve under blowing up the origin in $\mathbb{C}^2$. Inductively we define the functor $E_{(n)}$:

Definition 4.1. Let C be a plane curve singularity, and let $\overline{C}$ for $i = 1, \dots t$ be the connected components of the strict transform of the first blowing up of C (corresponding to the tangential components of C). Suppose that for $2 \leq j \leq n-1$ the functor $E_{(j)}$ has been defined, together with natural maps $E_{(j)} \longrightarrow E_{(j-1)}$. Then an element of $E_{(n)}(A)$ by definition is a $(\alpha; \overline{u}_1, \dots, \overline{u}_m)$ with

(1) $\alpha \in E_{(n-1)}(A)$. We let B_α be the A–space obtained by blowing up successively along the A–sections of α. We let $\alpha^* = (C_A, \overline{s}; \overline{s}_1, \dots, \overline{s}_t)$ be the image of α in $E_{(2)}$.

For each $j = 1, \dots, t$, we therefore have ordered collections α_j of A–sections in α such that $(\overline{C}_j, \alpha_j) \in E_{n-1\overline{C}_j}(A)$.

(2) $\overline{u}_i$ is an A–section of B_α.

(3) Every $\overline{u}_i$ lies over a section $\overline{s}_j$ of $B_{\overline{s}}$ in α^*; letting $\overline{u}_{j,1}, \dots, \overline{u}_{j,q}$ denote all such sections we have for all $j = 1, \dots, t$ that

$$(\overline{C}_j, \alpha_j; \overline{u}_{j,1}, \dots, \overline{u}_{j,q}) \in E_{n-1\overline{C}_j}(A).$$

Exactly as in Wahl 2.5 and 3.2 one shows that $E_{(n)}$ has a very good deformation theory, and that for $N \gg 0$ the natural maps $E_{(N+1)} \longrightarrow E_{(N)}$ is bijective. It is however *false* in general that for such a big N the natural morphism from $E_{(N)}$ to the functor of deformations of C is injective, contrary to the functor $E^{(N)}$.

Wahl defines the functor of equisingular deformation ES to be $E^{(N)}$ for $N \gg 0$. It is a subfunctor of the deformation functor of C. Similarly we define:

Definition 4.2. For a plane curve singularity we define the simultaneous resolution functor $\mathrm{SR} = \mathrm{SR}_C$ to be $E_{(N)}$ for $N \gg 0$.

The functor SR is in general not a subfunctor of the deformation functor of C. We now calculate the tangent space $\mathrm{SR}(\mathbb{C}[\varepsilon])$ of SR inductively. For this, consider the blowing up $(Y, E) \xrightarrow{p} (\mathbb{C}^2, 0)$. The strict transform of C under p split up into say t connected components, say $\overline{C}_1, \dots, \overline{C}_t$. We define $C_i = p(\overline{C}_i)$. We suppose that C is given by $f = 0$ for a square free f, and we let $f = f_1 \cdots f_t$, where C_i is given by $f_i = 0$. We let p_i be the intersection point of $\overline{C}_i$ with E.

We chose coordinates x, y on $\mathbb{C}^2$, so we get homogeneous coordinates $(u : v)$ on the exceptional projective line E. The blowing up is given by the equation $uy = xv$. For simplicity, we will assume that none of the p_i is the point with homogeneous coordinates $(0 : 1)$. The homogeneous coordinates of p_i we denote by $(1 : a_i)$. Thus we only have to look at the chart $u = 1$ in the blowing up. In these coordinates, the blowing up is given by $y = xv$, and thus the strict transform $\overline{C}$ is given by $\overline{f}(x, v) = 0$, where we have the equality $f(x, y) = f(x, xv) = x^m \overline{f}(x, v)$. Here m is the multiplicity of C. Similar for f_i and $\overline{f}_i$.

By induction we may suppose known the tangent space $\mathrm{SR}_{C_i,p_i}(\mathbb{C}[\varepsilon])$ for $i = 1, \dots, t$. For each element in $\mathrm{SR}_{C_i,p_i}(\mathbb{C}[\varepsilon])$ we get, in particular, an infinitesimal (equimultiple) deformation of (C_i, p_i) given, say, by $f_i + \varepsilon g$. Thus we have a natural map $\mathrm{SR}_{C_i,p_i}(\mathbb{C}[\varepsilon]) \longrightarrow \mathbb{C}\{x, v - a_i\}$, whose image we call J_i. If m_i is the multiplicity of C_i, we may assume that the curve C_i is given by a Weierstrass polynomial of degree m_i in v. Applying the Weierstraß Division Theorem, we may assume that the degree of g in v is smaller than m_i.

Now let $((f_i + \varepsilon\overline{h}_i), \alpha_i)$ be a element of $\mathrm{SR}_{C_i,p_i}(\mathbb{C}[\varepsilon])$. Thus α_i is an ordered system of infinitesimal sections, and we assume that $\overline{h}_i(x, v - a_i)$ has degree smaller than m_i in v. We consider $x^{m_i}\overline{h}_i(x, v - a_i)$, where m_i is the multiplicity of C_i. Using the equation $y = xv$, we can eliminate v, and get an element $h_i(x, y) \in \mathbb{C}\{x, y\}$. We consider the product $h_i' = h_i(x, y) \prod_{j \neq i} f_i$, and the sum $h' = \sum_{i=1}^t h_i'$. Together with the trivial section $\overline{s}$ we get the following element

$$((f + \varepsilon h'), \overline{s}, \alpha_1, \dots, \alpha_t)\,.$$

of $\mathrm{SR}(\mathbb{C}[\varepsilon])$. We thus get a map $\psi : \prod_i \mathrm{SR}_{C_i}(\mathbb{C}[\varepsilon]) \longrightarrow \mathrm{SR}_C(\mathbb{C}[\varepsilon])$. The following is obvious.

Lemma 4.3. *The image of ψ consist of all elements of $\mathrm{SR}(\mathbb{C}[\varepsilon])$ whose first infinitesimal section is trivial.*

Furthermore, we have the trivial deformations of C, which lie in the Jacobian ideal of f. They certainly resolve simultaneously. Those for which the first section is not trivial, have as image in E the infinitesimal deformations $\big((f + a\varepsilon\frac{\partial f}{\partial x} + b\varepsilon\frac{\partial f}{\partial y}), (x+a\varepsilon, x+b\varepsilon)\big)$ for $(a,b) \in \mathbb{C}^2\{(0,0)\}$. These lead to two elements α_x and α_y in $\mathrm{SR}(\mathbb{C}[\varepsilon])$, corresponding to the pairs $(a,b) = (1,0)$ and $(0,1)$.

Proposition 4.4. *For a plane curve singularity,* $\mathrm{SR}(\mathbb{C}[\varepsilon])$ *is generated by the image of* ψ*,* α_x *and* α_y*.*

Proof. Let an element α in $\mathrm{SR}(\mathbb{C}[\varepsilon])$ whose first section is given by the ideal $(x - a\varepsilon, y - b\varepsilon)$ for $(a,b) \in \mathbb{C}^2$. Then $\alpha + a\alpha_x + b\alpha_y$ has trivial first section, hence is contained in the image of ψ by the lemma. □

We can easily calculate generators of the image of ψ, and thus of $\mathrm{SR}(\mathbb{C}[\varepsilon])$ as follows. Namely, take generators of $\mathrm{SR}_{C_i,p_i}(\mathbb{C}[\varepsilon])$ such that their images in J_i (say under the map β) generate J_i as $\mathbb{C}\{x\}$–module. That is, in each degree $p < m_i$ we calculate all elements of J_i of degree p in v. As the polynomials in v of degree smaller than m_i is a finitely generated $\mathbb{C}\{x\}$–module, and J_i is a submodule, we can calculate such generators by computing a standard basis. The inverse image under β of such a generator is a finitely generated $\mathbb{C}\{x\}$–module. By induction one only has to show this for an infinitesimal equimultiple deformation. But the different sections form a two dimensional vector space, so certainly finitely generated $\mathbb{C}\{x\}$–module.

So take finitely many of those generators of $\mathrm{SR}_{C_i,p_i}(\mathbb{C}[\varepsilon])$ as $\mathbb{C}\{x\}$–module. Their images under ψ then generate the image of ψ.

Now we turn our attention to the calculation of $\mathrm{ES}(\mathbb{C}[\varepsilon])$. We consider the minimal good resolution (Z,F) of C, and we let $F_1, \ldots .F_p$ be the irreducible components of F.

Theorem 4.5. *There is a natural map of vector spaces*

$$\varphi : \mathrm{SR}(\mathbb{C}[\varepsilon]) \longrightarrow \oplus_{i=1}^{p} H^1(F_i, N_{F_i/Z})$$

whose kernel is $\mathrm{ES}(\mathbb{C}[\varepsilon])$*: the tangent space of the equisingular deformations of* C*.*

Proof. Clearly we have $\mathrm{ES}(\mathbb{C}[\varepsilon]) \subset \mathrm{SR}(\mathbb{C}[\varepsilon])$. Conversely, suppose α is an element of $\mathrm{SR}(\mathbb{C}[\varepsilon])$. Associated to it is a space B_α, which is obtained by blowing up $\mathbb{C}^2 \times \mathrm{Spec}(\mathbb{C}[\varepsilon])$ successively along the sections described by α. Hence we get an infinitesimal deformation of some resolution of the curve C, in particular a deformation of the minimal resolution of Z, that is an element of $H^1(Z, \Theta_Z)$. We look at the induced deformation of a small neighborhood of each F_i. More precisely, from the natural composition $\Theta_Z \longrightarrow \oplus\Theta_Z \otimes \mathcal{O}_{F_i} \longrightarrow \oplus N_{F_i/Z}$ we get a map $H^1(Z, \Theta_Z) \longrightarrow \oplus_{i=1}^{p} H^1(F_i, N_{F_i/Z})$. Thus gives the map φ. The kernel consists of elements of $\mathrm{SR}(\mathbb{C}[\varepsilon])$ which induce trivial deformations of all small neighborhoods of each F_i. This is equivalent to saying that we get an infinitesimal equisingular deformation of the minimal resolution Z, that is, an

element of $\mathrm{ES}_Z(\mathbb{C}[\varepsilon])$. By Wahl, [16], 5.7 we have a natural smooth functor $\mathrm{ES}_C \longrightarrow \mathrm{ES}_Z$, whose kernel on tangent spaces is given by the tangent space of the functor ES', which is the functor of all equisingular deformations of C for which all sections can be trivialized, see [16] 5.4. This proves the Theorem. $\square$

Remark 4.6. Note that the $\mathrm{ES}'(\mathbb{C}[\varepsilon])$, the tangent space of all equisingular deformations of C whose sections can be trivialized, certainly is in the kernel of φ. In this case, the resolution Z is deformed trivially. This $\mathrm{ES}'(\mathbb{C}[\varepsilon])$ is easy to compute, see Wahl [16] 6.3. We thus get an induced map

$$\varphi : \mathrm{SR}(\mathbb{C}[\varepsilon])/\,\mathrm{ES}'(\mathbb{C}[\varepsilon]) \longrightarrow \oplus_{i=1}^{p} H^1(F_i, N_{F_i/Z})$$

Given an element α in $\mathrm{SR}(\mathbb{C}[\varepsilon])$, one can write down the space B_α, which then gives the wanted infinitesimal deformation of Z. We have to see what the image in each $H^1(E, N_{E/Z})$ is for each exceptional curve E. We first recall how to describe $H^1(E, N_{E/Z})$ by means of Cech cohomology. In our case, the embedding E in Z will always be described in the following way. On E we have homogeneous coordinates $(u : v)$. The space Z is in a neighborhood of E, given by two charts.

- In the first chart U_1 we have coordinates v and x.
- In the second chart U_2 we have coordinates y and u.
- The transition functions on the intersection $U_1 \cap U_2$ are

  $$u = v^{-1}, \quad y = \alpha(v)x$$

  where $\alpha(v)$ is a polynomial in v of degree k (with zero constant coefficient).

The curve E is given by the zero set of $I := (x, y)$. In fact, if Z is obtained from the blowing up of $\mathbb{C}^2$, by blowing up points $(1 : a_1), \ldots, (1 : a_{k-1})$ (not necessarily distinct), and no other points on E, then the polynomial $\alpha(v)$ is equal to $v\prod_{i=1}^{k-1}(v - a_i)$.

The Cech cohomology group $H^1(E, N_{E/Z})$ is easily described. On the intersection $U_1 \cap U_2$, it is given by an element of $\mathrm{Hom}(I, \mathcal{O}/I)$ which sends y to $h(v)$, where $h(v)$ is a polynomial of degree smaller than k with no constant coefficient. (Note that $v = u^{-1}$ on $U_1 \cap U_2$.) If it sends y to an arbitrary holomorphic function $h(v)$, then the class in $H^1(E, N_{E/Z})$ is obtained by the remainder of $h(v)$ through $v\prod_{i=1}^{k-1}(v - a_i)$, and forgetting the constant term. The corresponding infinitesimal deformation of a small neighborhood of E in Z is given by the transition functions

$$u = v^{-1}, \quad y = \alpha(v)x + \varepsilon h(v).$$

This is the Kodaira-Spencer description of infinitesimal deformations. So in this way we can make the map φ explicit, and thus calculate the equisingularity ideal.

In practice it is probably better to proceed with induction: suppose known the $\mathrm{ES}_{\overline{C}_i}(\mathbb{C}[\varepsilon])$ for $i = 1, \ldots, t$, where the $\overline{C}_i$ are the connected components of the strict transform of C under blowing up the origin in $\mathbb{C}^2$. (Note that these can all be identified with ideals, that is, the sections are uniquely determined by the deformation.) Let E be the exceptional divisor, and suppose that on the minimal good resolution (Z, F) the curve E has selfintersection $-k$. That means, that we have to blow up $k-1$ times in the exceptional curve of the blowing up up $\mathbb{C}^2$ in the process to arrive at the minimal good resolution (Z, F). We consider the submodule $\mathrm{AES} := \psi(\prod_{i=1}^{t}\left(\mathrm{ES}_{\overline{C}_i}(\mathbb{C}[\varepsilon])\right) \subset \mathrm{SR}(\mathbb{C}[\varepsilon])$.

Lemma 4.7. *The map* $\mathrm{AES} \longrightarrow \oplus_{F_i \neq E} H^1(F_i, N_{F_i/Z})$ *induced by* φ *is the zero map.*

Proof. After blowing up in the trivial section, we get, by *construction* of AES equisingular deformations of all connected components of the strict transforms of C, which then, by Theorem 4.5, map to the zero element of $\oplus_{F_i \neq E} H^1(F_i, N_{F_i/Z})$. □

We therefore get an induced map

$$\varphi : \mathrm{AES} \longrightarrow H^1(E, N_{E/Z}),$$

whose kernel is equal to $\mathrm{ES}(\mathbb{C}[\varepsilon])$. Obviously $\mathcal{O}_C / \mathrm{ES}'(\mathbb{C}[\varepsilon])$ is a finite dimensional vector space. Thus we get an induced map of finite dimensional vector spaces:

$$\varphi : \mathrm{AES} / \mathrm{ES}'(\mathbb{C}[\varepsilon]) \longrightarrow H^1(E, N_{E/Z}).$$

Example 4.8. We take the curve singularity given by

$$f = (y^4 + x^5)^2 + x^{11} = 0.$$

The Tjurina number $\tau(f)$ one computes with SINGULAR [5] to be 55. The dimension of the equisingular stratum one computes to be 6. The total transform of the blowing up is given by $x^8\big((v^4 + x)^2 x^3 = 0$. We see that the strict transform given by $\overline{f} = (v^4 + x)^2 + x^3 = 0$ is an A_{11} singularity, whose equisingularity ideal is trivial, that is, generated by the partial derivatives of a defining equation. Thus it is generated as $\mathbb{C}\{x, v\}/(\overline{f})$–module by $\frac{\partial \overline{f}}{\partial x}$ and $\frac{\partial \overline{f}}{\partial v}$. We need however generators as $\mathbb{C}\{x\}$–module. $\mathbb{C}\{x, v\}/(\overline{f})$ is a free $\mathbb{C}\{x\}$–module of rank 8, generated by $1, v, \ldots, v^7$. Thus we have the following $\mathbb{C}\{x\}$–generators:

$$\begin{gathered}\frac{\partial \overline{f}}{\partial x}, v\frac{\partial \overline{f}}{\partial x}, \ldots, v^7\frac{\partial \overline{f}}{\partial x}, \\ \frac{\partial \overline{f}}{\partial v}, v\frac{\partial \overline{f}}{\partial v}, \ldots, v^7\frac{\partial \overline{f}}{\partial v}.\end{gathered}$$

We need classes of these elements in $\mathbb{C}\{x, v\}/(\overline{f})$ of degree smaller than eight in v. For these we can take

$$2(v^4 + x) + 3x^2, \ldots, 2v^3(v^4 + x) + 3v^3x^2$$

$$-2x(v^4+x+x^2)+3v^4x^2,\dots,v^3(-2x(v^4+x+x^2)+3v^4x^2),$$

$$v^3(v^4+x), xv^4+x^2+x^3,\dots,v^3(xv^4+x^2+x^3),$$

$$-x^2(v^4+x+x^2)+v^4x^3\dots,-x^2v^2(v^4+x+x^2)+v^6x^3.$$

We first look at $\frac{\partial \overline{f}}{\partial x} = 2(v^4+x)+3x^2$. The corresponding section is given by $(x+\varepsilon, v)$. Thus we blow up the trivial infinitesimal deformation of the first blowing up given by the transition functions

$$u = v^{-1}, \quad y = vx$$

in the section $(x+\varepsilon, v)$. In the appropriate chart this blowing up is given by the equation $x+\varepsilon = vx_1$. Hence after this blowing up, a neighborhood of E is given by the transition functions

$$u = v^{-1}, \quad y = v^2x_1 - \varepsilon v.$$

On the first blowing up, the equisingular deformation of the A_{11} singularity was given by $\overline{f}(x+\varepsilon, v) = 0$. If the second blowing up of the undeformed curve is given by $\overline{\overline{f}}(x_1, v) = 0$, we see that for the deformed one we have $\overline{\overline{f}}(x_1, v) = 0$. Hence we can continue blowing up in trivial sections, leading to the deformation of Z given in a neighborhood of E by the transition functions

$$u = v^{-1}, \quad y = v^5x_4 - \varepsilon v.$$

This gives a nontrivial element in $H^1(E, N_{E/Z})$.

Now look at $v\frac{\partial \overline{f}}{\partial x}$. The corresponding infinitesimal deformation of the strict transform of C is given by $\overline{f}(x+\varepsilon v, v) = 0$. We blow up in the trivial section (x, v) and in the appropriate chart we have $x = vx_1$, so we get

$$u = v^{-1}, \quad y = v^2x_1.$$

The induced deformation on this blowing up of C is described by $\overline{f}(vx_1+\varepsilon v, v) = v^2\overline{\overline{f}}(x_1+\varepsilon, v)$. Thus we can apply the previous discussion and blow up in $x_1+\varepsilon$ leading to the third blowing up given by $x_1+\varepsilon = vx_2$ getting the transformation functions

$$u = v^{-1}, \quad y = v^3x_2 - \varepsilon v^2.$$

We can now, as before, continue with trivial sections leading to an infinitesimal deformation of Z given in a neighborhood of E by the transition functions

$$u = v^{-1}, \quad y = v^5x_4 - \varepsilon v^2.$$

This again gives a nontrivial element in $H^1(E, N_{E/Z})$. In general one see that the deformation given by

$$\overline{f}(x+\varepsilon g(v), v)$$

for g of degree less than 4 leads to an infinitesimal deformation of the minimal resolution of Z which in a neighborhood of E is described by the transition functions

$$u = v^{-1}, \quad y = \alpha(v)w - \varepsilon v g(v).$$

For terms of degree greater or equal to 4, we get trivial deformations along E, thereby inducing infinitesimal equisingular deformations of C.

Thus for example $v^4 \frac{\partial \overline{f}}{\partial x} = -2xv^4 - 2x^2 - 2x^3 + 3x^2v^4$ gives the element $x^8(-2xv^4 - 2x^2 - 2x^3 + 3x^2v^4) = -2x^5y^4 - 2x^{10} - 2x^{11} + 3x^6y^4$. This gives an equisingular deformation. For

$$v^5 \frac{\partial \overline{f}}{\partial x}, \quad v^6 \frac{\partial \overline{f}}{\partial x}, \quad v^7 \frac{\partial \overline{f}}{\partial x},$$

we get

$$-2x^4y^5 - 2x^9y - 2x^{10}y + 3x^5y^5, -2x^3y^6 - 2x^8y^2 - 2x^9y^2 + 3x^4y^6,$$

$$-2x^2y^7 - 2x^7y^3 - 2x^8y^3 + 3x^3y^7.$$

Similarly one sees that the deformations corresponding to $\frac{\partial \overline{f}}{\partial v}, \ldots, v^7 \frac{\partial \overline{f}}{\partial v}$ all lead to equisingular deformations of C. These we see to map under ψ to

$$x\frac{\partial f}{\partial y},\ x^5y^4 + x^{10} + x^{11},\ x^4y^5 + x^9y + x^{10}y,\ x^3y^6 + x^8y^2 + x^9y^2,$$

$$x^2y^7 + x^7y^3 + x^8y^3,\ -x^6y^4 - x^{11} - x^{12} + x^7y^4,\ -x^5y^5 - x^{10}y - x^{11}y + x^6y^5,$$

$$-x^4y^6 - x^9y^2 - x^{10}y^2 + x^5y^6.$$

With SINGULAR we can now compute a standard basis for the equisingularity ideal of C (as $\mathbb{C}\{x, y\}$–module). The result is

$$y^7 + x^5y^3,\ 10x^4y^4 + 10x^9 + 11x^{10},\ x^3y^6 + x^8y^2 + x^9y^2,\ x^{11},\ x^{10}y,\ x^9y^2,\ x^8y^3,$$

and that indeed the equisingular stratum has dimension 6. The equisingular deformations of C all of whose sections are trivial one computes to be the ideal

$$(x^6, x^5y, x^3y^2, x^2y^3, xy^4, y^5)^2 \mod f.$$

Together with the trivial deformations these give only a five dimensional stratum. The "extra" infinitesimal equisingular deformation, is given by

$$(y^4 + x^5 + \varepsilon x^3y^2)^2 + x^{11}.$$

Remark 4.9. Luengo informed me that he has a different method for computing the equisingularity ideal. Furthermore, Campillo and Greuel seem to have a method for computing the equisingularity ideal by using the Hamburger Noether expansion. A more detailed study of the map φ leads to an another proof of Theorem 3.5.

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